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Axiomatizing the identities of binoid languages

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Abstract

We present a nontrivial axiomatization for the equational theory of binoid languages, the subsets of a free binoid. In doing so, we prove that a conjecture given in our previous paper was true: the identical laws of ordinary (string) languages, written separately using 'horizontal' and 'vertical' operation symbols form a required complete system of axioms.

1. Introduction

Among the many fruitful directions in formal language theory, recently there seems to be an increased interest in attempts to generalize the classical theory of string languages to two and more 'dimensions'. Naturally, it is more than reasonable to expect that many links and similarities to the classical theory will be found — and indeed, yet another such link will be discovered in this paper. On the other hand, it occurs quite often that, while investigating the combinatorial and algebraic features of these generalized words and languages, we find ourselves in complicated mathematical situations, without even a distant resemblance to those we know from our (daily) practice with strings and their collections.

For example, if one defines a *two-dimensional word*, or a *picture*, to be just a rectangular matrix of letters, such a definition leads to the theory of *two-dimensional* (or *picture*) *languages*, which represent one of the most popular ways of generalizing string languages. We direct the reader to the survey [8] of Giammarresi and Restivo, who present a thorough introduction to the theory of picture languages.

It is highly probable, however, that a formal language theorist who prefers an algebraic approach to his subject would like to think of words as elements of a free monoid Σ^*, rather than as plain discrete objects. This is, of course, not much of a difference in itself, but if we look just a bit closer, we will find a handful of reasons to make such a distinction, one of them being the classical interplay of semigroups and automata theory and the powerful mixture formed today by the theories of finite semigroups and formal languages. Hence, by taking such a course of consideration, we can change the underlying algebraic structure, and so, instead of the free monoid, consider the *free binoid* (i.e. a free bisemigroup with a unit, the free object in the variety of all algebras with *two* associative operations and a common identity element). A *binoid language* is now an arbitrary subset of a free binoid. Binoid

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languages instantly give rise to an algebraic setting, since the two operations of the free binoid induce two types of concatenation of languages ($\rightarrow$, $\downarrow$), as well as two kinds of iteration ($^{>}$, $^{\vee}$).

These notions were investigated, for example, in the fundamental study of Ésik and Németh [6], and also in [4,7,10,11]. The list of highly related references would include, among others, the work of Grabowski [9], Hoogenboom and ten Pas [12,13], Kuske [16,17], Lodaya and Weil [18–20]. Let us mention that binoid languages have numerous applications: for instance, they successfully model the rectangular tilings used in the layout of GUI elements.

The general question we are interested in here is that of identical laws satisfied by binoid languages, with respect to the above operations, expanded (as in the classical case) by the set-theoretic union $+$ and some constants. Our aim is to present a nontrivial axiomatization for such an equational theory. In the sequel, we denote this equational theory by Θ.

The question of identities satisfied by ordinary languages already belongs to what might be considered as 'textbook material', and it was truly put forward in the famous booklet [2] by John H. Conway. Back in the sixties, it was proved by Redko [22] that the equational theory of (string) languages (=pairs of rational expressions representing the same language) cannot be axiomatized by means of finitely many identities. However, it was only in the beginning of the nineties that anybody was able to present a nontrivial (necessarily infinite) equational axiomatization. Then, by putting in what must be credited as epic efforts, Krob [15] and Bloom and Ésik [1] managed (by entirely different methods) to confirm a long-standing conjecture from [2]. It turned out that these sought-after axiom systems are deeply involved with the theory of finite (simple) groups [5].

It was conjectured in [3] that the string language identities in the 'horizontal signature' ($\rightarrow$, $^{>}$, along with $+$ and the constants) together with the 'vertical' string language identities ($+$, $\downarrow$, $^{\vee}$, ...) form a complete axiom system for Θ. It is the goal of the present paper to prove that this conjecture is true. As a consequence, we eventually provide a finite axiomatization for Θ which involves implications and identities.

2. Preliminaries and the main result

Of course, before we launch into the main argument, it is necessary to present all the notions in a more precise fashion. We start by discussing representations of the free binoid on an alphabet Σ. It became usual (see, e.g., [4,6,7]) to consider its elements as *Σ-labelled biposets* of a special type. Namely, if $<_1$ and $<_2$ are strict partial orders on a set A, then the structure $\mathcal{A} = (A, <_1, <_2)$ is called a *biposet*. A Σ-labelled biposet is just a biposet $\mathcal{A}$ together with a labelling function $\lambda_{\mathcal{A}} : A \to \Sigma$.

A letter $x \in \Sigma$ can be regarded as a singleton poset S_x labelled by x, and new posets are obtained by two binary operations $\circ_1, \circ_2$, where $\mathcal{A} \circ_i \mathcal{B}$ $(i = 1, 2)$ is defined on $A \cup B$ by

$$<_j^{\mathcal{A}\circ_i\mathcal{B}} = \begin{cases} <_j^{\mathcal{A}} \cup <_j^{\mathcal{B}} & \text{if } j \neq i, \\ <_j^{\mathcal{A}} \cup <_j^{\mathcal{B}} \cup (A \times B) & \text{if } j = i. \end{cases}$$

A biposet is *series-parallel* (*sp* for short) if it is generated from the singletons by the two product operations. It was proved in [4] that all sp-biposets form a free bisemigroup on Σ, so that by adding the 'empty poset' ϵ we obtain the free binoid (freely) generated by Σ. Therefore, the elements of the free binoid over Σ can be identified with Σ-labelled sp-biposets.

However, we shall not adopt this way of handling free binoids. Instead, we would rather like to think about their elements as terms in the signature

$$T = \{\rightarrow, \downarrow, \epsilon\},$$

reduced with respect to the associativity of the two binary operations $\rightarrow, \downarrow$ (ϵ is the constant denoting the empty term). We feel that such an approach makes the subsequent definitions more clear and intuitive.

More precisely, we define a *bi-word* to be a finite $(T \cup \Sigma)$-labelled tree b (the empty tree ϵ included) such that the following conditions hold:

(1) the labels of all leaves belong to Σ,
(2) the labels of all non-leaves belong to $\{\rightarrow, \downarrow\}$, such that all vertices whose distance from the root is an even number are labelled by the same symbol as the root, while all other non-leaf vertices are labelled by the opposite symbol,
(3) each non-leaf has at least two successors.

In this way, the vertices on a given floor are either leaves, or have the same label from $\{\rightarrow, \downarrow\}$ (which depends on the parity of the floor).

The *depth* of b, $\delta(b)$, is the usual depth of the underlying tree of b, where the root is considered to be at depth 0. Also, we define $\delta(\epsilon) = 0$.

With respect to the label of the root, we distinguish between three kinds of bi-words.

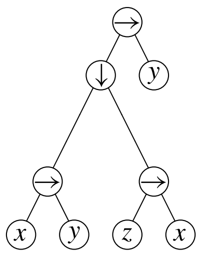
(1) A bi-word is *horizontal* if the label of its root is $\rightarrow$.
(2) A bi-word is *vertical* if the label of its root is $\downarrow$.
(3) All other bi-words, that is the empty one and the one-element trees (to be identified with letters), are called *neutral*.

The sets of horizontal, vertical and neutral bi-words over Σ we denote respectively by H_Σ, V_Σ and N_Σ (with $N_\Sigma^+ = N_\Sigma \setminus \{\epsilon\}$). The set of all bi-words over Σ is denoted by BW_Σ. Furthermore, $\mathsf{BW}_\Sigma^{\leqslant d}$ is the set of all elements of BW_Σ of depth $\leqslant d$. We set $X_\Sigma^{\leqslant d} = X_\Sigma \cap \mathsf{BW}_\Sigma^{\leqslant d}$ for $X \in \{H, V\}$.

Example 1. The bi-word

$$b(x, y, z) = ((x \rightarrow y) \downarrow (z \rightarrow x)) \rightarrow y$$

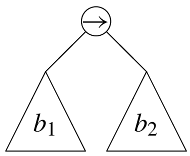
is represented by the following labelled tree:



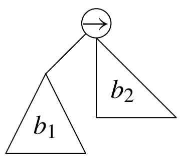
This is a horizontal bi-word with $\delta(b) = 3$.

The operations on bi-words are now defined in a fairly obvious way. For simplicity, we describe only $\rightarrow$, the definition of $\downarrow$ being analogous. We have three cases.

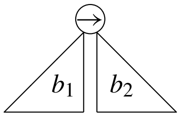
(1) Let $b_1, b_2 \in V_\Sigma \cup N_\Sigma$. Then $b_1 \rightarrow b_2$ is a tree with a root labelled by $\rightarrow$ having b_1 and b_2 as two successor subtrees.



(2) $b_1 \in V_\Sigma \cup N_\Sigma$ and $b_2 \in H_\Sigma$ (the 'reverse' case is similar). The root of $b_1 \rightarrow b_2$ is labelled by $\rightarrow$ and its successor subtrees are b_1 and the successor subtrees of the root of b_2 (in that order).



(3) If $b_1, b_2 \in H_\Sigma$, then $b_1 \rightarrow b_2$ is obtained by simply 'gluing together' b_1 and b_2 with a common root.



Hence, we may consider each horizontal (vertical) bi-word b of depth d as a $\rightarrow$-product ($\downarrow$-product) of vertical (horizontal) and neutral bi-words of depth at most $d-1$. Pushing this idea just a little bit further, every element of $H_\Sigma^{\leqslant d}$ ($V_\Sigma^{\leqslant d}$) is in fact a word (string) over the alphabet $V_\Sigma^{\leqslant(d-1)} \cup N_\Sigma^+$ ($H_\Sigma^{\leqslant(d-1)} \cup N_\Sigma^+$).

It is now a matter of a routine to show that $(\mathsf{BW}_\Sigma, \rightarrow, \downarrow, \epsilon)$ is a free binoid on Σ. The proof is omitted and left to the reader as an exercise (it suffices to map a bi-word $b(x_1, \ldots, x_n)$ to the sp-biposet $b(S_{x_1}, \ldots, S_{x_n})$).

A *binoid language* (we call it a *bi-language* for short) over Σ is just any subset L of BW_Σ. Besides the operation $+$ of the set-theoretic union, we have the horizontal product

$$L_1 \rightarrow L_2 = \{b_1 \rightarrow b_2 :\ b_1 \in L_1,\ b_2 \in L_2\},$$

the vertical product

$$L_1 \downarrow L_2 = \{b_1 \downarrow b_2 :\ b_1 \in L_1,\ b_2 \in L_2\},$$

and two kinds of closure: the horizontal iteration

$$L^{>} = \bigcup_{n\geqslant 0} L^{\overset{n}{\rightarrow}},$$

where $L^{\overset{0}{\rightarrow}} = \{\epsilon\}$, and for $n \geqslant 1$, $L^{\overset{n}{\rightarrow}} = L \rightarrow \cdots \rightarrow L$ ($n-1$ times $\rightarrow$), and the vertical iteration

$$L^{\vee} = \bigcup_{n\geqslant 0} L^{\downarrow n},$$

with similar conventions as above. Of course, we have the constants $\varnothing$ and $\{\epsilon\}$. In this way, we obtain the *algebra of bi-languages* over Σ:

$$\mathbf{BiLang}_\Sigma = (\mathcal{P}(\mathsf{BW}_\Sigma), +, \rightarrow, \downarrow, {}^{>}, {}^{\vee}, \varnothing, \{\epsilon\}).$$

The *equational theory of bi-languages* is the set Θ of all identities which are true in all bi-language algebras. The result of this paper is now as follows.

Theorem 1. *Let Γ_1 be the set of all identities of string languages in the 'horizontal' signature $\{+, \rightarrow, {}^{>}, \varnothing, \epsilon\}$, while Γ_2 is the same set of identities in the 'vertical' signature $\{+, \downarrow, {}^{\vee}, \varnothing, \epsilon\}$. Then $\Gamma_1 \cup \Gamma_2$ is a complete set of axioms for Θ.*

This theorem solves Problem 1 from [3] and confirms a conjecture that follows the formulation of that problem.

3. Birational expressions and identities

Birational expressions are the 'binoid analogues' of (ordinary) rational expressions. More precisely, a birational expression is simply a term in the signature $\{+, \rightarrow, \downarrow, {}^{>}, {}^{\vee}, \varnothing, \epsilon\}$ of bi-language algebras. Of course, rational expressions in signatures $\{+, \rightarrow, {}^{>}, \varnothing, \epsilon\}$ and $\{+, \downarrow, {}^{\vee}, \varnothing, \epsilon\}$ are also (special) birational expressions; these will be called $\rightarrow$*-rational* and $\downarrow$*-rational expressions*, respectively.

By means of the standard evaluation of letters, $x \mapsto \{x\}$ for all $x \in \Sigma$, each birational expression $\alpha(x_1, \ldots, x_n)$ has its *value* $\mathcal{B}(\alpha) \subseteq \mathsf{BW}_\Sigma$. Bi-languages of the form $\mathcal{B}(\alpha)$ are called *birational*. It is known (cf. Theorem 4.7 of [6]) that for each birational bi-language L there is a finite d such that $\mathcal{B}(\alpha) \subseteq \mathsf{BW}_\Sigma^{\leqslant d}$. The least such d is called the *depth*, $\delta(\alpha)$, of the expression α.

A birational expression α is *horizontal* (*vertical*) if $\mathcal{B}(\alpha) \subseteq H_\Sigma \cup N_\Sigma$ ($\mathcal{B}(\alpha) \subseteq V_\Sigma \cup N_\Sigma$). An important observation is now that each birational expression may be split into its horizontal and vertical 'component' by using the valid identities between $\rightarrow$- and $\downarrow$-rational expressions.

Lemma 2 (*Decomposition Lemma*). *Let Γ_1 and Γ_2 be as in Theorem 1, and let α be a birational expression. Then there exist birational expressions α^h and α^v such that*

$$\alpha = \alpha^h + \alpha^v$$

follows from $\Gamma_1 \cup \Gamma_2$ and α^h, α^v are horizontal and vertical expressions, respectively. Furthermore, α^h and α^v can be chosen to be 'maximal' in the sense that $\mathcal{B}(\alpha^h) = \mathcal{B}(\alpha) \cap (H_\Sigma \cup N_\Sigma)$ and $\mathcal{B}(\alpha^v) = \mathcal{B}(\alpha) \cap (V_\Sigma \cup N_\Sigma)$.

Proof. We prove the lemma by induction on the complexity of α.

If α is an atomic expression ($\varnothing$, ϵ, or some $a \in \Sigma$), then simply $\alpha = \alpha + \alpha$ is the identity of the desired form, following trivially from $\Gamma_1 \cup \Gamma_2$.

The case $\alpha \equiv \beta + \gamma$ (where $\equiv$ stands for the *graphical* equality of terms) is also very easy. By assumption, the identities $\beta = \beta^h + \beta^v$ and $\gamma = \gamma^h + \gamma^v$ with the required properties follow from $\Gamma_1 \cup \Gamma_2$, and now an application of the associative law for $+$ suffices to obtain $\alpha = (\beta^h + \gamma^h) + (\beta^v + \gamma^v)$. So, it remains to set $\alpha^h \equiv \beta^h + \gamma^h$ and $\alpha^v \equiv \beta^v + \gamma^v$. It is not hard to see that the 'maximality' requirements are also met in this way.

Assume $\alpha \equiv \beta \to \gamma$ (the case $\alpha \equiv \beta \downarrow \gamma$ is analogous). We also assume that $\mathcal{B}(\beta), \mathcal{B}(\gamma) \neq \varnothing$, for otherwise $\mathcal{B}(\alpha) = \varnothing$. Therefore, $\alpha = \varnothing$ is a valid identity, whence the following claim resolves the situation.

Claim 1. *Any valid identity of the form $\alpha = \varnothing$ is provable from $\Gamma_1 \cup \Gamma_2$.*

This claim is quite easily verified by induction on the complexity of α, so its proof is omitted. Needless to say, $\alpha = \varnothing + \varnothing$ is the desired identity in this case.

Now, at first glance, one might say that α is a horizontal expression, so that something like $\alpha^h \equiv \alpha$ and $\alpha^v \equiv \varnothing$ would suffice. However, this is not entirely correct: each of the bi-languages $\mathcal{B}(\beta), \mathcal{B}(\gamma)$ can contain the empty tree ϵ, and thus if e.g. $b_1 = \epsilon \in \mathcal{B}(\beta)$ and $b_2 \in \mathcal{B}(\gamma)$, then $b_1 \to b_2 = b_2$ inherits the label of its root from b_2 (on which we have no constraints at all). Hence, we need

Claim 2. *Let β be a birational expression such that $\epsilon \in \mathcal{B}(\beta)$. Then there exists an expression $\bar{\beta}$ such that $\epsilon \notin \mathcal{B}(\bar{\beta})$ and the identity $\beta = \epsilon + \bar{\beta}$ follows from $\Gamma_1 \cup \Gamma_2$.*

Proof of Claim 2. We prove our claim by induction on the complexity of β. If $\beta \equiv \lambda$, then $\bar{\beta} \equiv \varnothing$ is a suitable choice. Otherwise, if $\beta \equiv \beta_1 + \beta_2$, then $\epsilon \in \mathcal{B}(\beta_i)$ for at least one index $i \in \{1, 2\}$. For such i, an identity of the form $\beta_i = \epsilon + \bar{\beta}_i$ is provable from $\Gamma_1 \cup \Gamma_2$. For an illustration, assume that this is the case for $i = 1$, but not for $i = 2$, the other cases being similar. Then we obtain the identity

$$\beta = \epsilon + \bar{\beta}_1 + \beta_2,$$

and it suffices to set $\bar{\beta} \equiv \bar{\beta}_1 + \beta_2$. As the next step, let $\beta \equiv \beta_1 \to \beta_2$ (for $\beta \equiv \beta_1 \downarrow \beta_2$ we proceed in an analogous fashion). Then $\epsilon \in \mathcal{B}(\beta_1) \cap \mathcal{B}(\beta_2)$, and so by induction the identities $\beta_i = \epsilon + \bar{\beta}_i$ are provable from $\Gamma_1 \cup \Gamma_2$ for $i = 1, 2$. Hence, we obtain

$$\beta = (\epsilon + \bar{\beta}_1) \to (\epsilon + \bar{\beta}_2) = \epsilon + \bar{\beta}_1 + \bar{\beta}_2 + (\bar{\beta}_1 \to \bar{\beta}_2),$$

so that $\bar{\beta} \equiv \bar{\beta}_1 + \bar{\beta}_2 + (\bar{\beta}_1 \to \bar{\beta}_2)$ will do. Finally, let $\beta \equiv \beta_1^{>}$ (the case $\beta \equiv \beta_1^{\vee}$ is analogous). By induction, we have the identity $\beta_1 = \epsilon + \bar{\beta}_1$ at our disposal, and thus we deduce

$$\beta = \bar{\beta}_1^{>} = \epsilon + (\bar{\beta}_1 \to \bar{\beta}_1^{>}),$$

and by putting $\bar{\beta} \equiv \bar{\beta}_1 \to \bar{\beta}_1^{>}$, we finish the proof of the claim. ♡

So, in the sequel, assume that $\mathcal{B}(\beta)$ contains ϵ, while $\mathcal{B}(\gamma)$ does not, the other possibilities being handled similarly. Then we may deduce an identity $\beta = \epsilon + \bar{\beta}$ as in the above claim, and so we further obtain

$$\alpha = (\epsilon + \bar{\beta}) \to \gamma = \gamma + (\bar{\beta} \to \gamma).$$

Now it can be correctly claimed that $\bar{\beta} \to \gamma$ is a horizontal expression (moreover, a 'purely' horizontal one, whose value contains no neutral bi-words), while by the inductive assumption we decompose $\gamma = \gamma^h + \gamma^v$. Therefore, it suffices to put $\alpha^h \equiv \gamma^h + (\bar{\beta} \to \gamma)$ and $\alpha^v \equiv \gamma^v$. The 'maximality' condition is verified easily.

Finally, it remains to consider the case when α is obtained from a simpler expression by means of an iteration, for example a horizontal one, $\alpha \equiv \beta^{>}$. Then we deduce

$$\alpha = \epsilon + \beta + (\beta \to \beta \to \beta^{>}).$$

If the value of β does not contain ϵ, then $\beta \to \beta \to \beta^{>}$ is a (purely) horizontal expression. Otherwise, make use of Claim 2 to obtain $\beta = \epsilon + \bar{\beta}$. We have

$$\begin{aligned}\beta \to \beta \to \beta^{>} &= (\epsilon + \bar{\beta}) \to (\epsilon + \bar{\beta}) \to \bar{\beta}^{>} = (\epsilon + \bar{\beta} + (\bar{\beta} \to \bar{\beta}))\bar{\beta}^{\vee} \\ &= \bar{\beta}^{\vee} = \epsilon + \bar{\beta} + (\bar{\beta} \to \bar{\beta} \to \bar{\beta}^{>}) = \beta + (\bar{\beta} \to \bar{\beta} \to \bar{\beta}^{>}),\end{aligned}$$

which results in

$$\alpha = \epsilon + \beta + (\bar{\beta} \to \bar{\beta} \to \bar{\beta}^{>}).$$

The identity $\beta = \beta^h + \beta^v$ (deducible by the inductive hypothesis) allows us to set $\alpha^v \equiv \epsilon + \beta^v$ and $\alpha^h \equiv \epsilon + \beta^h + (\bar{\beta} \to \bar{\beta} \to \bar{\beta}^{>})$ (or without bars if $\epsilon \notin \mathcal{B}(\beta)$), thus completing our proof. $\square$

The following simple, yet key observation is that each identity of bi-languages 'splits' into a horizontal and vertical one. Recall that the fact that $\alpha_1 = \alpha_2$ holds in all bi-language algebras is equivalent to $\mathcal{B}(\alpha_1) = \mathcal{B}(\alpha_2)$ (see Theorem 5 of [3]).

Lemma 3. *Let α_1, α_2 be birational expressions, and let α_i^h, α_i^v, $i = 1, 2$, be as in the above lemma. The identity $\alpha_1 = \alpha_2$ belongs to Θ if and only if both $\alpha_1^h = \alpha_2^h$ and $\alpha_1^v = \alpha_2^v$ belong to Θ.*

Proof. The converse implication ($\Leftarrow$) holds by the very definition of the birational expressions involved. On the other hand, we have $\mathcal{B}(\alpha_i^h) = \mathcal{B}(\alpha_i) \cap (H_\Sigma \cup N_\Sigma)$ and $\mathcal{B}(\alpha_i^v) = \mathcal{B}(\alpha_i) \cap (V_\Sigma \cup N_\Sigma)$ for $i = 1, 2$. Therefore, $\mathcal{B}(\alpha_1) = \mathcal{B}(\alpha_2)$ implies $\mathcal{B}(\alpha_1^h) = \mathcal{B}(\alpha_2^h)$ and $\mathcal{B}(\alpha_1^v) = \mathcal{B}(\alpha_2^v)$, whence the conclusion follows. $\square$

As a consequence of the above lemma, we may without any loss of generality assume that any identity $\alpha_1 = \alpha_2$ under consideration is such that both α_1 and α_2 are, for example, horizontal expressions (the vertical case is dual). If all such identities from Θ turn out to be provable from $\Gamma_1 \cup \Gamma_2$, then the previous two lemmata would finish off the proof of our Theorem 1.

To avoid further complications arising e.g. from the fact that if α is, say, a horizontal expression, then $\alpha \downarrow \epsilon$ is also horizontal – in spite of having the form $\alpha \downarrow \beta$ – we call a birational expression *trimmed* if it is either graphically equal to $\varnothing$, or it has no subterm equivalent either to $\varnothing$ or ϵ, except, possibly, for a single summand graphically equal to ϵ. We shall see that each expression can be 'cleaned up' so that it becomes trimmed. The following lemma is basically a generalization of the Claim 1 from the proof of Lemma 2.

Lemma 4. *Let α be a birational expression. Then there is a trimmed expression α_0 such that $\alpha = \alpha_0$ follows from $\Gamma_1 \cup \Gamma_2$.*

Proof. The proof consists of a standard inductive argument on the the complexity of α. It takes only a short reflection to see that one needs to employ only the following identities:

$$\begin{aligned}&\varnothing + x = x + \varnothing = x,\\ &\varnothing \to x = x \to \varnothing = \varnothing \downarrow x = x \downarrow \varnothing = \varnothing,\\ &\epsilon \to x = x \to \epsilon = \epsilon \downarrow x = x \downarrow \epsilon = x,\\ &\varnothing^{>} = \varnothing^{\vee} = \epsilon = \epsilon + \epsilon,\\ &(\epsilon + x)^{>} = x^{>},\\ &(\epsilon + x)^{\vee} = x^{\vee},\end{aligned}$$

supplemented by the distributivity laws for $\to, \downarrow$ over $+$. $\square$

Therefore, in the remainder of the paper, we assume that all the involved birational expressions are trimmed.

It is now time to start preparing the way for the main concept that will lead us, once properly defined, directly to the required proof. This is the concept of the *adjoined string identity* of a (horizontal) identity $\alpha_1 = \alpha_2$, where α_1, α_2 are (horizontal) birational expressions.

We call a $\rightarrow$-rational ($\downarrow$-rational, birational) expression α *linear* if each of its variables (letters) occurs in it exactly once.

Lemma 5 (*Linearization Lemma*). *Let α be a horizontal birational expression.*

(i) *There exist a linear $\rightarrow$-rational expression $\alpha'(x_1, \ldots, x_n)$ and vertical expressions $\beta_1, \ldots, \beta_n$ such that*

$$\alpha \equiv \alpha'(\beta_1, \ldots, \beta_n).$$

In such a case, if $\delta(\alpha) \geqslant 1$, we have $\delta(\alpha) = \max(\delta(\beta_1), \ldots, \delta(\beta_n)) + 1$.

(ii) *There exists a horizontal birational expression $\hat{\alpha}$, a linear $\rightarrow$-rational expression $\alpha''(x_1, \ldots, x_k)$ and vertical expressions $\beta'_1, \ldots, \beta'_k$ such that*
  (a) *the identity $\alpha = \hat{\alpha}$ follows from $\Gamma_1 \cup \Gamma_2$,*
  (b) *$\hat{\alpha} \equiv \alpha''(\beta'_1, \ldots, \beta'_k)$, and*
  (c) *$\epsilon \notin \mathcal{B}(\beta'_i)$ and $\mathcal{B}(\beta'_i) \neq \varnothing$ for all $1 \leqslant i \leqslant k$.*

Proof. (i) As several times above, we use induction on the complexity of α.

If α is an atomic expression, then $\alpha' \equiv \alpha$ suffices, and if α is a letter, then $n = 1$ and $\beta_1 \equiv \alpha$. Of course, now we have $\delta(\alpha) = 0$, so there is nothing left to prove.

If $\alpha \equiv \alpha_1 + \alpha_2$, then both α_1, α_2 are horizontal expressions, and thus we apply the induction hypothesis: there are linear $\rightarrow$-rational expressions $\alpha'_1(x_1, \ldots, x_p)$ and $\alpha'_2(x_1, \ldots, x_q)$, and vertical expressions $\beta_1, \ldots, \beta_p, \gamma_1, \ldots, \gamma_q$ such that

$$\begin{aligned} \alpha_1 &\equiv \alpha'_1(\beta_1, \ldots, \beta_p), \\ \alpha_2 &\equiv \alpha'_2(\gamma_1, \ldots, \gamma_q), \end{aligned}$$

and

$$\begin{aligned} \delta(\alpha_1) &= \max(\delta(\beta_1), \ldots, \delta(\beta_p)) + 1, \\ \delta(\alpha_2) &= \max(\delta(\gamma_1), \ldots, \delta(\gamma_q)) + 1, \end{aligned}$$

provided $\delta(\alpha_1), \delta(\alpha_2) > 0$. Then it is sufficient to rename $\beta_{p+i} \equiv \gamma_i$, $1 \leqslant i \leqslant q$, and to define α' to be $\alpha'_1(x_1, \ldots, x_p) + \alpha'_2(x_{p+1}, \ldots, x_{p+q})$, where we assume that $x_1, \ldots, x_{p+q}$ are distinct letters. Furthermore,

$$\delta(\alpha) = \max(\delta(\alpha_1), \delta(\alpha_2)),$$

so that if both of the above formulae (for $\delta(\alpha_1)$ and $\delta(\alpha_2)$) are correct, we obtain

$$\delta(\alpha) = \max(\delta(\beta_1), \ldots, \delta(\beta_{p+q})) + 1.$$

But the above conclusion still holds if, for example, $\delta(\alpha_2) = 0$: for then we have $\delta(\beta_{p+i}) = 0$ for $1 \leqslant i \leqslant q$.

If $\alpha \equiv \alpha_1 \rightarrow \alpha_2$, then there are essentially three different possibilities. If both α_1, α_2 are horizontal, we proceed in a completely analogous manner as in the above paragraph (replacing, of course, the role of $+$ by that of $\rightarrow$). It takes only a few moments to check that all the conclusions made there still hold true. The other subcase is when both α_1, α_2 are vertical expressions. Then we define

$$\alpha'(x, y) \equiv x \rightarrow y$$

and $\beta_1 \equiv \alpha_1$, $\beta_2 \equiv \alpha_2$. As α is assumed to be trimmed, we may take for granted that both $\mathcal{B}(\alpha_1), \mathcal{B}(\alpha_2)$ contain a nonempty bi-word. Hence, $\delta(\alpha) = \max(\delta(\alpha_1), \delta(\alpha_2)) + 1$. Finally, the third possibility is that precisely one of α_1, α_2 is not horizontal (say, α_2). This subcase is something of an 'intermediate' between the previous two. Let $\alpha_1 \equiv \alpha'_1(\beta_1, \ldots, \beta_p)$, all the objects having the same meaning as in the previous paragraph. Define

$$\alpha' \equiv \alpha'_1(x_1, \ldots, x_p) \rightarrow x_{p+1}.$$

Clearly, $\alpha \equiv \alpha'(\beta_1, \ldots, \beta_p, \alpha_2)$ and $\delta(\alpha) = \max(\delta(\alpha_1), \delta(\alpha_2) + 1)$. Therefore, $\delta(\alpha) = \max(\delta(\beta_1), \ldots, \delta(\beta_p), \delta(\alpha_2)) + 1$.

The case $\alpha \equiv \alpha_1^>$ is very much similar to the previous one: it should be distinguished between the subcases when α_1 is horizontal and vertical. In the latter case,

$$\alpha'(x) \equiv x^>$$

and $\beta_1 \equiv \alpha_1$ is a suitable choice. Since α is trimmed, there is a nonempty bi-word in $\mathcal{B}(\alpha_1)$, implying $\delta(\alpha) = \delta(\alpha_1)+1$. On the other hand, if α_1 is horizontal, we may apply the inductive assumption, and so – by retaining the earlier notation – we set

$$\alpha' \equiv (\alpha_1'(x_1, \ldots, x_p))^>.$$

The depths of α and α_1 must coincide in this case, and so we are done.

Finally, it remains to consider the cases $\alpha \equiv \alpha_1 \downarrow \alpha_2$ and $\alpha \equiv \alpha_1^\vee$. However, they are impossible, since α is trimmed. Indeed, it is instantly clear that the expression $\alpha_1 \downarrow \alpha_2$ can be horizontal if and only if one of α_1, α_2 is equivalent either to $\varnothing$, or to λ. Similarly, $\alpha_1^\vee$ is horizontal if and only if the same conclusion holds for α_1. Trimmed expressions exclude such possibilities, so the proof of the first part of the lemma is complete.

(ii) Suppose that we have $\epsilon \in \mathcal{B}(\beta_i)$ for some $1 \leqslant i \leqslant n$ as a result of the above inductive process. Then we may use Claim 2 (from the proof of Decomposition Lemma) for all such i to obtain identities of the form $\beta_i = \epsilon + \bar{\beta}_i$ where $\epsilon \notin \mathcal{B}(\bar{\beta}_i)$. Let

$$x_i^\epsilon = \begin{cases} x_i & \epsilon \notin \mathcal{B}(\beta_i), \\ \epsilon & \mathcal{B}(\beta_i) = \{\epsilon\}, \\ \epsilon + x_i & \text{otherwise}, \end{cases} \qquad \beta_i^\epsilon = \begin{cases} \beta_i & \epsilon \notin \mathcal{B}(\beta_i), \\ \epsilon & \mathcal{B}(\beta_i) = \{\epsilon\}, \\ \epsilon + \bar{\beta}_i & \text{otherwise}, \end{cases}$$

and let $\alpha''(x_{i_1}, \ldots, x_{i_k})$, $i_1 < \cdots < i_k$, be the expression obtained by trimming

$$\alpha'(x_1^\epsilon, \ldots, x_n^\epsilon).$$

Lemma 4 implies that $\alpha''(x_{i_1}, \ldots, x_{i_k}) = \alpha'(x_1^\epsilon, \ldots, x_n^\epsilon)$ follows from $\Gamma_1 \cup \Gamma_2$, as well as $\beta_i = \beta_i^\epsilon$ for all $1 \leqslant i \leqslant n$. Moreover, if $\hat{\beta}_i$ denotes $\bar{\beta}_i$ if $\epsilon \in \mathcal{B}(\beta_i)$ and β_i otherwise, then the substitution of $\hat{\beta}_i$ in x_i^ϵ produces β_i^ϵ, so that the identities

$$\alpha''(\hat{\beta}_{i_1}, \ldots, \hat{\beta}_{i_k}) = \alpha'(\beta_1^\epsilon, \ldots, \beta_n^\epsilon) = \alpha'(\beta_1, \ldots, \beta_n) \equiv \alpha$$

also follow from $\Gamma_1 \cup \Gamma_2$. Therefore, it suffices to define $\beta_j' \equiv \hat{\beta}_{i_j}$ for $1 \leqslant j \leqslant k$ and $\hat{\alpha} \equiv \alpha''(\hat{\beta}_{i_1}, \ldots, \hat{\beta}_{i_k})$. By doing so, the requirements (a) and (b) are automatically satisfied. Also, (c) holds since by construction we have $\epsilon \notin \mathcal{B}(\hat{\beta}_{i_j}) \neq \varnothing$ for all $1 \leqslant j \leqslant k$. $\square$

We do not pretend that the above representation of a horizontal expression as a $\rightarrow$-rational expression applied to vertical expressions of lesser depth is unique; indeed, due to the possible presence of neutral bi-words in values of horizontal expressions, it is easy to construct counterexamples to the claim of uniqueness (e.g. $x + y$ can be viewed as a vertical expression β_1, or as $\beta_1 + \beta_2$, where $\beta_1 \equiv x$ and $\beta_2 \equiv y$). As a consequence, the adjoined string identity (introduced below) will not be an entity which is uniquely attached to a birational identity. Instead, one such identity can have many adjoined ones — however, this ambiguity will have no importance whatsoever.

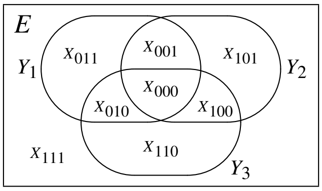
So, consider two horizontal birational expressions α_1, α_2 of positive depths. By the above Linearization Lemma, there exist $\rightarrow$-rational expressions α_1'', α_2'', non-negative integers $n < m$, and vertical expressions $\beta_1, \ldots, \beta_n, \beta_{n+1}, \ldots, \beta_m$ such that $\delta(\beta_i) < d = \max(\delta(\alpha_1), \delta(\alpha_2))$ and $\epsilon \notin \mathcal{B}(\beta_i) \neq \varnothing$ for all $i \leqslant m$, and

$$\begin{aligned} \alpha_1 &= \alpha_1''(\beta_1, \ldots, \beta_n), \\ \alpha_2 &= \alpha_2''(\beta_{n+1}, \ldots, \beta_m), \end{aligned}$$

follow from $\Gamma_1 \cup \Gamma_2$. For all $i \leqslant m$, let $Y_i = \mathcal{B}(\beta_i)$. These bi-languages form a family consisting of m nonempty subsets of $E = V_\Sigma^{\leqslant(d-1)} \cup N_\Sigma$. For $j \in \{0, 1\}$, define Y_i^j so that $Y_i^0 = Y_i$ and $Y_i^1 = E \setminus Y_i$. Furthermore, for a binary sequence $\sigma \in \{0, 1\}^m$, let

$$X_\sigma = \bigcap_{i=1}^m Y_i^{\sigma(i)}.$$

What we have done here is in fact building up the standard partition of E determined by its subsets Y_i, $1 \leqslant i \leqslant m$ (for $m = 3$, this is illustrated by the following familiar picture).



Definition (*Adjoined String Identity*). Let α_1, α_2 be horizontal birational expressions of depth > 0, and let α_1'', α_2'', m, Y_i^j ($1 \leqslant i \leqslant m$, $j \in \{0, 1\}$) and X_σ ($\sigma \in \{0, 1\}^m$) be as above. Define the sets $\Lambda_i \subseteq \{0, 1\}^m$, $1 \leqslant i \leqslant m$, such that

$$\sigma \in \Lambda_i \quad \text{if and only if} \quad \sigma(i) = 0 \text{ and } X_\sigma \neq \varnothing.$$

We say that the identity (in the 'horizontal' signature)

$$\alpha_1''\left(\sum_{\sigma\in\Lambda_1} x_\sigma, \ldots, \sum_{\sigma\in\Lambda_n} x_\sigma\right) = \alpha_2''\left(\sum_{\sigma\in\Lambda_{n+1}} x_\sigma, \ldots, \sum_{\sigma\in\Lambda_m} x_\sigma\right)$$

is an *adjoined string identity* (or *doppelgänger*) for the identity $\alpha_1 = \alpha_2$. Note that both sides of this identity are $\rightarrow$-rational expressions, and so it is basically an identity which may or may not be true for ordinary, string languages. The alphabet for this identity is $\Xi_m = \{x_\sigma : \sigma \in \{0, 1\}^m\}$ (actually, the letter corresponding to the sequence of all 1's is never used, by the very definition of the sets Λ_i).

Example 2. Consider the identity

$$x^> + (x^\vee)^> = (x^\vee)^>$$

which is easily seen to be a valid one. It can be considered as

$$\beta_1^> + \beta_2^> = \beta_3^>,$$

where $\beta_1 \equiv x$ and $\beta_2 \equiv \beta_3 \equiv x^\vee$. Now, we should get rid of ϵ from $\mathcal{B}(\beta_2) = \mathcal{B}(\beta_3)$, so we make use of $x^\vee = \epsilon + x \downarrow x^\vee$ and proceed with $x \downarrow x^\vee$ instead of $x^\vee$. (Although, this is not really necessary in this example, as the transformation in (ii) of the Linearization Lemma has as its sole purpose to make sure that the case $X_\sigma = \{\epsilon\}$ cannot occur. However, this will turn out to be essential in the proof of Lemma 6 below.) For the corresponding values we obviously have $Y_1 \subset Y_2 = Y_3$, thus $\Lambda_1 = \{000\}$ and $\Lambda_2 = \Lambda_3 = \{000, 100\}$. If, for simplicity, we write x for x_{000} and y for x_{100}, our adjoined string identity becomes

$$x^> + (x + y)^> = (x + y)^>,$$

a familiar law telling us that the Kleene star is monotone.

The patient reader might agree that this is the proper time and place for asking the rhetorical question: what is the connection (if any) between an identity and its string doppelgänger? Let us rush to the answer immediately.

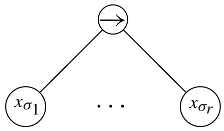
4. Proof of Theorem 1

Lemma 6 (*Doppelgänger Lemma*). *Let α_1, α_2 be birational expressions such that the identity $\alpha_1 = \alpha_2$ belongs to Θ (i.e. it holds in bi-language algebras). Then its adjoined string identity is a valid one (i.e. it belongs to Γ_1).*

Proof. For simplicity, let $\mu_1 = \mu_2$ be an adjoined string identity for $\alpha_1 = \alpha_2$, where μ_1, μ_2 are $\rightarrow$-rational expressions as above. Assume that $\mu_1 = \mu_2$ is not valid; then there exists a bi-word w which, for example, belongs to $\mathcal{B}(\mu_1)$, but not to $\mathcal{B}(\mu_2)$. In fact, we may consider w as a word and $\mathcal{B}(\mu_k)$, $k = 1, 2$, as string languages, by interpreting the horizontal product as concatenation. By the construction of a doppelgänger, there is an integer m such that $\mu_1 = \mu_2$ is an identity over the alphabet Ξ_m. Thus, w is of the form

$$w \equiv x_{\sigma_1} \to \cdots \to x_{\sigma_r},$$

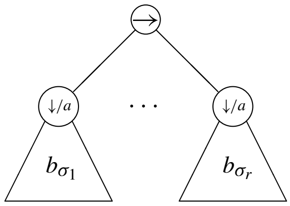
for some (not necessarily distinct) $\sigma_1, \ldots, \sigma_r \in \{0, 1\}^m$.



For each $\sigma \in I = \{0, 1\}^m \setminus \{11\ldots1\}$ such that $X_\sigma \neq \varnothing$ (where the sets X_σ are determined as in the above definition with respect to $\alpha_1 = \alpha_2$), choose a bi-word $b_\sigma \in X_\sigma$ in an arbitrary fashion, and let

$$\widetilde{w} \equiv b_{\sigma_1} \to \cdots \to b_{\sigma_r}.$$

This bi-word (depicted below) is well-defined since $w \in \mathcal{B}(\mu_1)$ implies that for each p ($1 \leqslant p \leqslant r$) there is an i_p ($1 \leqslant i_p \leqslant m$) such that $\sigma_p \in \Lambda_{i_p}$. Therefore, we necessarily have $X_{\sigma_p} \neq \varnothing$, by the definition of the sets Λ_i. Also, we have $b_\sigma \not\equiv \epsilon$, since by the construction of a doppelgänger, $\epsilon \notin Y_i$ for all $i \leqslant m$, and thus $\epsilon \notin X_\sigma$ for all $\sigma \in I$.



We claim that $\widetilde{w} \in \mathcal{B}(\alpha_1) \setminus \mathcal{B}(\alpha_2)$, from which the lemma follows immediately. Actually, it suffices to show that, regardless of the form of a word w, we have $w \in \mathcal{B}(\mu_k)$ if and only if $\widetilde{w} \in \mathcal{B}(\alpha_k)$.

The converse implication ($\Leftarrow$) will easily follow once we prove

Claim 3. *Let M_k, $k = 1, 2$, be the bi-language obtained from μ_k by evaluating x_σ as X_σ, $\sigma \in I$. Then $M_k = \mathcal{B}(\alpha_k)$.*

Proof of Claim 3. For each i, $1 \leqslant i \leqslant m$, we have

$$Y_i = \bigcup_{\sigma(i)=0} X_\sigma.$$

However, some of the X_σ's may be empty, and this is precisely what is recorded by the sets Λ_i. Hence, the above equality may be written as

$$Y_i = \bigcup_{\sigma \in \Lambda_i} X_\sigma.$$

This implies that we have

$$M_k = \alpha'_k(Y_{1+(k-1)n}, \ldots, Y_{m+(k-1)n}).$$

By recalling that $Y_j = \mathcal{B}(\beta_j)$ for any j and the definition of α'_k, the claim immediately follows. ♡

Now, by a direct application of Proposition 4 from [3] (with μ_k in the role of α and X_σ's instead of $L_1, \ldots, L_n$), we obtain the equality

$$M_k = \bigcup_{b \in \mathcal{B}(\mu_k)} b(X_\sigma)_{\sigma \in I},$$

where $b(X_\sigma)_{\sigma \in I}$ denotes the bi-language obtained by applying the bi-word b to the X_σ's. So, if $w \in \mathcal{B}(\mu_k)$, then $w(X_\sigma)_{\sigma \in I} \subseteq M_k$. However, by construction, $\widetilde{w} \in w(X_\sigma)_{\sigma \in I}$, thus $\widetilde{w} \in M_k = \mathcal{B}(\alpha_k)$.

(⇒) Assume that $\widetilde{w} \in \mathcal{B}(\alpha_k)$. By the above formula for M_k, we conclude that $\widetilde{w} \in b(X_\sigma)_{\sigma \in I}$ for some ($\rightarrow$-)word $b \in \mathcal{B}(\mu_k)$. We will show that this is possible if and only if $b \equiv w$; as a consequence, we will obtain $w \in \mathcal{B}(\mu_k)$, and the proof will be over.

From the given assumptions we have that there are bi-words $c_\sigma \in X_\sigma, \sigma \in I$, such that $\widetilde{w}$ is identical to the bi-word b' obtained from b by the substitution $x_\sigma \mapsto c_\sigma$. Recall that, by definition, all these bi-words c_σ are *vertical*; also, $c_\sigma \not\equiv \epsilon$ as $\epsilon \notin X_\sigma$ for all $\sigma \in I$. Thus, if

$$b \equiv x_{\tau_1} \rightarrow \cdots \rightarrow x_{\tau_s},$$

the representation

$$c_{\tau_1} \rightarrow \cdots \rightarrow c_{\tau_s}$$

is in fact the maximal decomposition of b' into $\rightarrow$-irreducible factors (cf. p. 44 of [3]). On the other hand, the same is true for the decomposition

$$b_{\sigma_1} \rightarrow \cdots \rightarrow b_{\sigma_r}$$

of $\widetilde{w}$. As $\widetilde{w} \equiv b'$, we have $r = s$ and $b_{\sigma_q} \equiv c_{\tau_q}$ for all $1 \leqslant q \leqslant r$. However, $b_{\sigma_q} \in X_{\sigma_q}$, while $c_{\tau_q} \in X_{\tau_q}$. Since the sets X_σ are disjoint, it follows that $\sigma_q = \tau_q$ for all $1 \leqslant q \leqslant r$. Hence, $w \equiv b$, as desired. □

The above lemma is the key ingredient for the induction step in the proof of our main result. Also, below it will become evident that the converse is also true: the validity of an adjoined string identity implies that the original identity itself must be valid.

Proof of Theorem 1. For birational expressions α_1, α_2, we define the *depth* of an identity $\alpha_1 = \alpha_2$ as $\max(\delta(\alpha_1), \delta(\alpha_2))$. We are going to prove by induction on the depth d of an identity from Θ that it can be deduced from $\Gamma_1 \cup \Gamma_2$. Of course, if $\alpha_1 = \alpha_2$ belongs to Θ, then $\delta(\alpha_1) = \delta(\alpha_2) = d$.

As already noted, we may assume that both α_1, α_2 are horizontal (or vertical) expressions; otherwise the identities $\alpha_1 = \alpha_1^h + \alpha_1^v$ and $\alpha_2 = \alpha_2^h + \alpha_2^v$ (as in Decomposition Lemma) can be proved from $\Gamma_1 \cup \Gamma_2$, and we may as well proceed with proving $\alpha_1^h = \alpha_2^h$ and $\alpha_1^v = \alpha_2^v$, the two being valid identities if and only if $\alpha_1 = \alpha_2$ is such, by Lemma 3.

If $d \leqslant 1$, then the identity we are dealing with is in fact a (horizontal) string identity, thus it belongs to Γ_1. Therefore, we assume that $d > 1$. As described in the Linearization Lemma, there are horizontal birational expressions $\hat{\alpha}_1, \hat{\alpha}_2$ such that $\alpha_1 = \hat{\alpha}_1$ and $\alpha_2 = \hat{\alpha}_2$ are consequences of $\Gamma_1 \cup \Gamma_2$, while the identity $\hat{\alpha}_1 = \hat{\alpha}_2$ has the form

$$\alpha_1''(\beta_1', \ldots, \beta_k') = \alpha_2''(\beta_{k+1}', \ldots, \beta_m'),$$

where α_1'', α_2'' are linear $\rightarrow$-rational expressions (involved later in the course of forming a doppelgänger identity), and $\beta_1', \ldots, \beta_m'$ are vertical expressions, all of them having depth at most $d - 1$, whose values $Y_1, \ldots, Y_m$ satisfy $\epsilon \notin Y_i \neq \varnothing$, $1 \leqslant i \leqslant m$. Let $\Lambda_1, \ldots, \Lambda_m$ and $X_\sigma, \sigma \in I$, be as in the definition of an adjoint string identity. We have already argued that

$$Y_i = \bigcup_{\sigma \in \Lambda_i} X_\sigma$$

holds for all $1 \leqslant i \leqslant m$.

Now recall the definition of the sets X_σ: these are intersections of birational bi-languages of the form Y_i and their relative complements $E \setminus Y_i$, where $E = V_\Sigma^{\leqslant(d-1)} \cup N_\Sigma$. Note that E is a birational bi-language, too (provided Σ is finite, but that is automatically true if Σ is taken to be the set of letters occurring in the identity $\alpha_1 = \alpha_2$): namely, E is the value of the expression

$$\left(\sum_{x \in \Sigma} x\right)^{\cdots > \vee},$$

where the vertical iteration $^\vee$ and the horizontal iteration $^>$ alternate $d - 1$ times. From Corollary 4.5 and Theorem 4.7 of [6] it follows that birational bi-languages are closed for the Boolean operations; that is to say, for intersections and relative complements (set differences). Thus $E \setminus Y_i$ is birational, as well as each X_σ. In addition, $X_\sigma \subseteq E$ for all

$\sigma \in I$. Therefore, there exists (for each $\sigma \in I$) a vertical birational expression ξ_σ of depth at most $d-1$ such that $X_\sigma = \mathcal{B}(\xi_\sigma)$. Furthermore, for each $1 \leqslant i \leqslant m$, the following vertical identity is valid:

$$\beta_i' = \sum_{\sigma \in \Lambda_i} \xi_\sigma . \tag{$*$}$$

But notice that each of the above m identities has depth $\leqslant d-1$, and so, by the induction hypothesis, they can be deduced from $\Gamma_1 \cup \Gamma_2$. On the other hand, by the Doppelgänger Lemma, the identity

$$\alpha_1''\left(\sum_{\sigma \in \Lambda_1} x_\sigma, \ldots, \sum_{\sigma \in \Lambda_n} x_\sigma\right) = \alpha_2''\left(\sum_{\sigma \in \Lambda_{n+1}} x_\sigma, \ldots, \sum_{\sigma \in \Lambda_m} x_\sigma\right)$$

is a valid one, thus it belongs to Γ_1. By applying the substitution $x_\sigma \mapsto \xi_\sigma$ and combining the resulting identity with $(*)$, we obtain a formal proof for $\alpha_1 = \alpha_2$, as required. $\square$

Remark 1. As was proved in [3], an identity in the signature $\{+, \rightarrow, \downarrow, ^{>}, ^{\vee}, \varnothing, \epsilon\}$ holds for all picture languages (where $\rightarrow$ and $\downarrow$ are interpreted as the column and the row product of picture languages, respectively, and $^{>}$, $^{\vee}$ as the corresponding iterations, see [8]) if and only if it belongs to Θ. Therefore, Theorem 1 also provides an equational axiomatization for the class of all picture language algebras $\mathbf{Pict}_\Sigma$ defined in [3].

Remark 2. Note that our Theorem 1 is similar in spirit to a result of Matz [21]. In Theorem 2 of that paper, Matz proved that a picture language L over a one-letter alphabet is representable by a birational expression (where a letter x is interpreted as the singleton language containing the 1×1 picture $[x]$, and operation symbols are interpreted as in the previous remark) if and only if L is a finite union of Cartesian products of ultimately periodic string languages. So, birational expressions are not powerful enough to capture any nontrivial relation between the horizontal and the vertical structure of two-dimensional languages, even in the one-letter case.

5. Some examples and a consequence

Example 3. As already pointed out in Example 2, one of the doppelgängers of the identity $x^{>} + (x^{\vee})^{>} = (x^{\vee})^{>}$ (transformed into $x^{>} + (x \downarrow x^{\vee})^{>} = (x \downarrow x^{\vee})^{>}$) is

$$x^{>} + (x+y)^{>} = (x+y)^{>}. \tag{$**$}$$

The nonempty X_σ's are $X_{000} = \{x\}$ and $X_{100} = \{x \downarrow x, x \downarrow x \downarrow x, \ldots\}$, which are represented by the birational expressions $\xi_{000} \equiv x$ and $\xi_{100} \equiv x \downarrow x \downarrow x^{\vee}$, respectively. Hence, the above proof gives that the considered identity follows from $(**)$ and

$$x + x \downarrow x \downarrow x^{\vee} = x \downarrow x^{\vee},$$

both being valid string identities in the corresponding signatures.

Example 4. Consider the identity

$$\begin{aligned}(x \rightarrow (y \rightarrow x)^{>})^{\vee} = {} & \left[\epsilon + (x \rightarrow y)^{>} \rightarrow x\right] \downarrow \\ & \downarrow \left[((x \rightarrow y)^{>} \rightarrow x) \downarrow (x \rightarrow (y \rightarrow x)^{>})\right]^{\vee}.\end{aligned}$$

We may note that the right-hand side of this identity is not trimmed, so our identity must be rewritten (using Lemma 4) as

$$\begin{aligned}(x \rightarrow (y \rightarrow x)^{>})^{\vee} = {} & \left[((x \rightarrow y)^{>} \rightarrow x) \downarrow (x \rightarrow (y \rightarrow x)^{>})\right]^{\vee} \\ & + ((x \rightarrow y)^{>} \rightarrow x) \downarrow \\ & \quad \downarrow \left[((x \rightarrow y)^{>} \rightarrow x) \downarrow (x \rightarrow (y \rightarrow x)^{>})\right]^{\vee}.\end{aligned}$$

Now, Lemma 5 gives

$$\beta_1^{\vee} = (\beta_3 \downarrow \beta_4)^{\vee} + \beta_5 \downarrow (\beta_6 \downarrow \beta_7)^{\vee},$$

where $\beta_1, \beta_3, \beta_7$ coincide with $x \to (y \to x)^>$, while $\beta_2, \beta_4, \beta_5, \beta_6$ coincide with $(x \to y)^> \to x$. Clearly, all these expressions represent the same bi-language (not containing ϵ), so that $\Lambda_i = \{0000\}$ for all $1 \leqslant i \leqslant 7$. Therefore, the corresponding doppelgänger is

$$a^\vee = (a \downarrow a)^\vee + a \downarrow (a \downarrow a)^\vee,$$

the well-known identity associated with the cyclic group $\mathbb{Z}_2$. The latter identity, along with $x \to (y \to x)^> = (x \to y)^> \to x$, implies the considered one.

Example 5. Our final example illustrates the necessity of the transformation described in (ii) of the Linearization Lemma. For instance, consider the identity

$$(x^\vee)^> = (x \downarrow x^\vee)^>,$$

which is obviously valid, since it is a consequence of $a^\vee = \epsilon + a \downarrow a^\vee$ and $a^> = (\epsilon + a)^>$. If we linearize this identity as

$$\beta_1^> = \beta_2^>,$$

where $\beta_1 \equiv x^\vee$ and $\beta_2 \equiv x \downarrow x^\vee$, the result is $X_{00} = \{x, x \downarrow x, \ldots\}$ and $X_{01} = \{\epsilon\}$, and we end up with the following doppelgänger:

$$(a + b)^> = a^>,$$

which is clearly false. Hence, we first have to replace β_1 by $\epsilon + x \downarrow x^\vee$, then trim the left-hand side, whence our identity turns into a trivial one.

We finish the paper by emphasizing one consequence of our Theorem 1 which appears to be important. Namely, it is well known that, although no finite equational axiomatization exists for identities of string languages, there are still some very short and elegant *implicational* axiomatizations. For example, we may single out a result due to Kozen [14] and Krob [15]: if $x \leqslant y$ is simply a short-hand for the identity $x + y = y$, then the axioms of additively idempotent semirings (semirings in which $x + x = x$ holds), together with

$$1 + xx^* \leqslant x^*,$$

and the implications (quasi-identities)

$$\begin{aligned} ax + b \leqslant x &\Rightarrow a^*b \leqslant x, \\ xa + b \leqslant x &\Rightarrow ba^* \leqslant x, \end{aligned}$$

are complete for the equational theory of string languages, i.e. the latter equational theory is precisely the set of all identities that are consequences of the above formulae (Krob proved in [15] that the second implication is actually redundant). Therefore, a straightforward application of Theorem 1 yields the following result.

Corollary 7. *The semiring axioms in the signatures* $\{+, \to, \varnothing, \epsilon\}$ *and* $\{+, \downarrow, \varnothing, \epsilon\}$*, the identities*

$$\begin{aligned} x + x &= x, \\ \epsilon + (x \to x^>) &\leqslant x^>, \\ \epsilon + (x \downarrow x^\vee) &\leqslant x^\vee, \end{aligned}$$

and the implications

$$\begin{aligned} (a \to x) + b \leqslant x &\Rightarrow a^> \to b \leqslant x, \\ (a \downarrow x) + b \leqslant x &\Rightarrow a^\vee \downarrow b \leqslant x, \end{aligned}$$

are complete for the equational theory Θ.

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